

DEFICIENCY SYMMETRIES OF SURGERIES IN S^3

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ABSTRACT. We examine certain symmetries in the deficiencies of a rational surgery on a knot in S^3 by comparing the Spin^c -structures on the rational surgery with those on a related integral surgery. We then provide an application of these symmetries in the form of a theorem that obstructs Dehn surgeries in S^3 . This last part unifies and generalises theorems by Greene underlying his work on the cabling conjecture, the lens space realisation problem, and the unknotting number of alternating 3-braids.

1. INTRODUCTION

In [3], Greene laid the basics for the follow beautiful theorem. He proved that if a lens space $L(p', q')$ (using the convention of $-p'/q'$ -surgery on the unknot) is obtained by p' -surgery on a knot $C \subset S^3$, then there exists an integral matrix A such that

$$-AA^t = Q_X \oplus (-p'),$$

where

- (1) Q_X is the adjacency matrix of the linear graph with weights $-b_i$ appearing in the Hirzebruch-Jung continued fraction $p'/q' = [b_1, \dots, b_\ell]$ (i.e. Q_X represents the intersection form of the graph's corresponding plumbed 4-manifold X); and
- (2) The entries of the final row of A form a *changemaker set* (i.e. a collection of non-negative integers σ_i with the property that, given coins worth σ_i , one can make up any value from zero to their sum).

By combining this theorem with some ingenious combinatorics, Greene was able to achieve spectacular success resolving the long-standing lens space realisation problem [4]. A similar theorem, involving the double branched cover of an alternating 3-braid K and half-integral surgeries, also allowed him to classify the K of this type with unknotting number one [2].

The main idea in this paper is to take the two “changemaker” theorems above and unify them in one. To this end, we will largely be concerned with the *deficiencies* of a $-p/q$ -surgery on a knot C in S^3 , where $p, q > 0$ are coprime. These objects are defined as the differences

$$D_C^{p/q}(\mathfrak{t}) := d(S_{-p/q}^3(C), \mathfrak{t}) - d(S_{-p/q}^3(U), \mathfrak{t}),$$

where U is the unknot, \mathfrak{t} a Spin^c -structure, and d the correction term of Ozsváth and Szabó (see [6]). Since this implicitly requires a bijection

$$\text{Spin}^c(S_{-p/q}^3(C)) \longleftrightarrow \text{Spin}^c(S_{-p/q}^3(U)),$$

we stipulate that the one used here is the standard one from the literature [2, 3, 4, 8]. When it is clear what we mean, we may drop the C from $D_C^{p/q}$.

In order to prove our result, we begin by establishing certain symmetries among the $D_C^{p/q}(i)$. Explicitly, we have the following theorem, where $p = nq - r$ and $0 < r < q$.

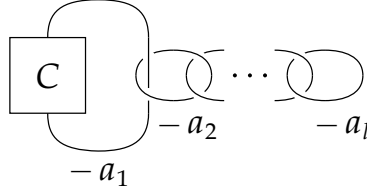


FIGURE 1. A Kirby diagram for $Y = S^3_{-p/q}(C)$, where $p/q = [a_1, \dots, a_\ell]$.

Theorem 1.1. *Let C be a knot in S^3 , and let $p, q > 0$ be coprime. Then there is a function $\mathfrak{r} : \text{Spin}^c(S^3_{-p/q}(C)) \rightarrow \text{Spin}^c(S^3_{-n}(C))$ such that the following diagram commutes:*

$$\begin{array}{ccc} \text{Spin}^c(S^3_{-p/q}(C)) & \xrightarrow{\mathfrak{r}} & \text{Spin}^c(S^3_{-n}(C)) \\ & \searrow D^{p/q} & \downarrow D^n \\ & & \mathbb{Q} \end{array}$$

and the fibres of \mathfrak{r} are of size q , with one exception over an element of $\text{Spin}^c(S^3_{-n}(C))$ which minimises the value of D^n . In particular, conjugation on $\text{Spin}^c(S^3_{-n}(C))$, under which D^n is invariant, lifts to a function on $\text{Spin}^c(S^3_{-p/q}(C))$ under which $D^{p/q}$ is invariant.

Stated thus, the theorem is in fact not difficult to prove, though the exhibition of an \mathfrak{r} requires considerably more effort. We provide one example towards the middle of the paper, before using it to generalise the two changemaker theorems. We let $p/q = [a_1, \dots, a_\ell]$, where $a_i \geq 2$ for $i \geq 2$ and $n = a_1$. It is not difficult to prove that such an expansion always exists.

Theorem 1.2. *Suppose that $Y = S^3_{-p/q}(C)$ for some knot $C \subset S^3$ and coprime $p, q > 0$, that W is the trace of the corresponding integral surgery in Figure 1, and that $-Y$ bounds a sharp, simply connected, negative-definite smooth 4-manifold X with intersection form Q_X and free $H_2(X)$. Then if*

$$d(Y, i) - d(S^3_{-p/q}(U), i) = 0$$

for either (a) one value of i if n is odd; or (b) $q - r + 1$ values of i if n is even, there exists an integral matrix A such that

$$-AA^t = Q_X \oplus Q_W.$$

In addition, if $q \neq 1$, one can choose A so that its last ℓ rows have the form

$$\begin{pmatrix} \sigma_r & \dots & \sigma_1 & 1 & 0 & & & \\ & & & -1 & 1 & \dots & 1 & \\ & & & & & \ddots & & \\ & & & & & & -1 & 1 & \dots & 1 & 0 \\ & & & & & & & -1 & 1 & \dots & 1 \end{pmatrix},$$

where there are exactly a_i non-zero entries in row $i = 2, \dots, \ell$, all ± 1 as above, and $\{\sigma_i\}_{i=1}^r$ forms a changemaker set. If, on the other hand, $q = 1$, then the last row of A can be chosen to have the form

$$(\sigma'_r \dots \sigma'_1 \sigma'_0),$$

where $\{\sigma'_i\}_{i=0}^r$ forms a changemaker set.

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2. AN APPLICATION OF KNOT FLOER HOMOLOGY

If $C \subset S^3$ is a knot, then recall that the associated knot Floer chain complex $CFK := CFK^\infty(S^3, C)$ is the \mathbb{Z} -module generated by a set X together with a filtration $\mathcal{I} : X \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ satisfying the properties

- (1) $\mathcal{I}(U \cdot x) = (i - 1, j - 1)$ if $\mathcal{I}(x) = (i, j)$; and
- (2) $\mathcal{I}(y) \leq \mathcal{I}(x)$ for all y with non-zero coefficient in ∂x .

Let S be a subset of $\mathbb{Z} \oplus \mathbb{Z}$ such that $(i, j) \in S$ implies $(i + 1, j), (i, j + 1) \in S$, and define $CFK\{S\}$ to be the quotient of the knot Floer complex by the submodule generated by those $x \in X$ with $\mathcal{I}(x) \in S$. In this notation, we let

$$A_k^+ := CFK\{i \geq 0 \text{ or } j \geq k\} \quad \text{and} \quad B_k^+ := CFK\{i \geq 0\},$$

where $k \in \mathbb{Z}$. As per [9], these complexes come equipped with canonical U -equivariant chain maps

$$v_k^+, h_k^+ : A_k^+ \longrightarrow B_k^+$$

such that v_k^+ is projection onto $CFK\{i \geq 0\}$ and h_k^+ is a composition of projection onto $CFK\{j \geq k\}$, identification with $CFK\{j \geq 0\}$, and chain homotopy equivalence with $CFK\{i \geq 0\}$. At sufficiently high gradings, these maps are isomorphisms and hence behave as multiplication by U^{V_k} and U^{H_k} respectively where $V_k, H_k \geq 0$ are integers. The following lemma is taken from [5].

Lemma 2.1. *The V_i and H_i satisfy the following properties:*

- (1) $V_0 = H_0$, and all $V_i, H_i \geq 0$;
- (2) The V_i are a non-increasing sequence, while the H_i are a non-decreasing sequence.

Using the labelling of Spin^c -structures given in Section 7 of [9], Ni and Wu proved the following proposition about $D_C^{p/q}(\mathfrak{t})$. It can be found in [5] as Proposition 2.11, though as stated here we have applied it to \overline{C} .

Proposition 2.2 (Ni-Wu). *Let C be any knot in S^3 , and let $p, q > 0$ be coprime. Then*

$$D_C^{p/q}(\mathfrak{t}_i) = 2 \max \left\{ V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor} \right\}.$$

As it stands, the labelling \mathfrak{t}_i used above is a difficult one to manipulate with general rational surgeries, but simplifies considerably in the case of an integral n -surgery. In this instance, the Spin^c -structure \mathfrak{t}_i is the one that admits an extension \mathfrak{s} over the cobordism $S^3 \rightarrow S_n^3(C)$ which satisfies

$$\langle c_1(\mathfrak{s}), [F] \rangle \equiv n + 2i \pmod{2n},$$

where F is a Seifert surface glued to the core of the attached handle. This fact will be useful to us later in our proofs.

We can now give a proof of the following result.

Lemma 2.3. *Let C be a knot in S^3 . Then*

$$\sum_{\mathfrak{t} \in \text{Spin}^c(S_{-p/q}^3(C))} D_C^{p/q}(\mathfrak{t}) = q \cdot \sum_{\mathfrak{t} \in \text{Spin}^c(S_{-n}^3(C))} D_C^n(\mathfrak{t}) - r \cdot \min_{\mathfrak{t} \in \text{Spin}^c(S_{-n}^3(C))} \{D_C^n(\mathfrak{t})\}.$$

Proof. We consider the integral surgery first. By a direct application of Proposition 2.2 we obtain

$$\sum_{\mathbf{t} \in \text{Spin}^c(S_{-n}^3(C))} D_C^n(\mathbf{t}) = 2 \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\}. \quad (1)$$

Our goal is to compare this with the rational surgery.

Labelling the Spin^c -structures on the rational surgery as \mathbf{t}_{iq+j} , we have the following bounds:

- (1) j ranges from 0 to $q-1$;
- (2) i ranges from 0 to $n-1$ if $j < q-r$, or from 0 to $n-2$ if $j \geq q-r$.

Rephrasing the second of these, i ranges from 0 to $n-1-\delta(j)$, where $\delta(j) := \left\lfloor \frac{j+r}{q} \right\rfloor$.

Consequently, using Proposition 2.2,

$$\sum_{j=0}^{q-1} \sum_{i=0}^{n-1-\delta(j)} D_C^{p/q}(\mathbf{t}_{iq+j}) = 2 \sum_{j=0}^{q-1} \sum_{i=0}^{n-1-\delta(j)} \max\{V_i, H_{i-n+\delta(j)}\}. \quad (2)$$

We fix j and observe that

$$\sum_{i=0}^{n-1-\delta(j)} \max\{V_i, H_{i-n+\delta(j)}\} = \begin{cases} \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} & \text{if } \delta(j) = 0 \\ \sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} & \text{if } \delta(j) = 1 \end{cases}. \quad (3)$$

Clearly, if $\delta(j) = 0$, then the RHS is the same as the RHS of (1). This happens for the first $q-r$ values of j , meaning that the situations of interest are the r larger cases when $\delta(j) = 1$. In effect, in order to obtain our result we need to establish that

$$\sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} = \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} - m,$$

where $2m$ is the value of the minimum deficiency.

Suppose that i' is chosen to be the largest integer such that the integral deficiency in Spin^c -structure $\mathbf{t}_{i'}$ is minimal. That is, that $\max\{V_{i'}, H_{i'-n}\}$ is minimal. Then there are two possibilities.

- (1) Suppose that $V_{i'} \geq H_{i'-n}$. As V_* is non-increasing and H_* is non-decreasing, it follows that $V_i \geq H_{i-n}$ for all $i \leq i'$, and hence that $V_i \geq V_{i+1} \geq H_{i-n+1}$ for all $i < i'$.

Going in the other direction, suppose that $V_{i'+1} > H_{i'-n+1}$. Then our choice of i' implies that $V_{i'+1} = \max\{V_{i'+1}, H_{i'-n+1}\} > \max\{V_{i'}, H_{i'-n}\} = V_{i'}$, a contradiction to the non-increasing behaviour of V_* . Thus $H_{i'-n+1} \geq V_{i'+1}$, and $H_{i-n+1} \geq H_{i-n} \geq V_i$ for all $i > i'$.

In the case $i = i'$, observe that

$$V_{i'} = \max\{V_{i'}, H_{i'-n}\} < \max\{V_{i'+1}, H_{i'-n+1}\} = H_{i'-n+1}.$$

Putting this together with the conclusions of the previous two paragraphs, we deduce that

$$\begin{aligned} \sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} &= \sum_{i=0}^{i'-1} V_i + \sum_{i=i'}^{n-2} H_{i-n+1} \\ &= \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} - V_{i'}. \end{aligned}$$

Observe that $V_{i'} = m$.

- (2) Suppose instead that $H_{i'-n} \geq V_{i'}$. This case is similar in nature, though a little more complicated: we define j' to be the smallest integer such that $\max\{V_k, H_{k-n}\}$ is minimal for $j' \leq k \leq i'$, and end with the conclusion

$$\begin{aligned} \sum_{i=0}^{n-2} \max\{V_i, H_{i-n+1}\} &= \sum_{i=0}^{j'-1} V_i + \sum_{i=j'}^{i'-1} H_{i'-n} + \sum_{i=i'}^{n-2} H_{i-n+1} \\ &= \sum_{i=0}^{n-1} \max\{V_i, H_{i-n}\} - H_{i'-n} \end{aligned}$$

and the observation that $H_{i'-n} = m$, by definition of i' .

To complete the proof, one puts the above information into (2) via (3) and compares with (1). \square

If one reads this argument carefully, one will find that it can be modified slightly to give a proof of Theorem 1.1. However, since this modified argument provides no insight as to the nature of \mathfrak{r} without a deeper knowledge of the labelling \mathfrak{t}_i , it is of limited use to us.

In light of the above lemma, a natural question at this point is: Which elements of $\text{Spin}^c(S_{-n}^3(C))$ minimise D^n ? The answer is below.

Lemma 2.4. *According to the parity of n ,*

- (1) *If n is even, then $\mathfrak{t}_{\frac{n}{2}}$ realises the minimal deficiency; and*
- (2) *If instead n is odd, then $\mathfrak{t}_{\frac{n+1}{2}}$ do the same.*

Proof. Recall that \mathfrak{t}_i evaluates, modulo $2n$ to $n + 2i$. Hence \mathfrak{t}_i and \mathfrak{t}_{n-i} are conjugates, and so, by the conjugation symmetry of correction terms, if $i \neq 0$ and \mathfrak{t}_i realises the minimum, so does \mathfrak{t}_{n-i} . Assuming that $i \leq n - i$, we claim that the same is true for all \mathfrak{t}_j with $i \leq j \leq n - i$. Indeed, let the minimum deficiency be $2m$, so that $\max\{V_i, H_{i-n}\} = \max\{V_{n-i}, H_{-i}\} = m$. We know that $m \geq V_i \geq V_j$ and $m \geq H_{-i} \geq H_{j-n}$, so $\max\{V_j, H_{j-n}\} \leq m$, and as m is minimal it follows that we have equality. Consequently, \mathfrak{t}_j also realises the minimum.

Thus, if \mathfrak{t}_i realises the minimum for $i \neq 0$, so do the \mathfrak{t}_j for the centralmost values of j , namely $\frac{n}{2}$ or $\frac{n+1}{2}$, depending on parity. The other possibility, of course, is that \mathfrak{t}_0 realises the minimum. In this case, observe that $D_C^n(\mathfrak{t}_0) = 2 \max\{V_0, H_{-n}\}$. By Lemma 2.1, $V_0 = H_0 \geq H_{-n}$, and we see that the deficiency is $2V_0$. Since $V_i \leq V_0$ and $H_{i-n} \leq H_0 = V_0$,

$$D_C^n(\mathfrak{t}_i) = 2 \max\{V_i, H_{i-n}\} \leq 2V_0 = D_C^n(\mathfrak{t}_0),$$

and \mathfrak{t}_0 is in fact the Spin^c -structure with the maximal deficiency. \square

3. PRELIMINARIES TO THE PROOFS

Our goal now is to exhibit a function $\mathfrak{r} : \text{Spin}^c(S_{-p/q}^3(C)) \rightarrow \text{Spin}^c(S_{-n}^3(C))$ that satisfies Theorem 1.1. This will require some enumeration of the Spin^c -structures on $S_{-p/q}^3(C)$, but because of the bijection

$$\text{Spin}^c(S_{-p/q}^3(C)) \longleftrightarrow \text{Spin}^c(S_{-p/q}^3(U))$$

described in Section 1, this enumeration need only consider the case $C = U$.



FIGURE 2. A weighted graph G with weighted adjacency matrix Q . Vertices are labelled v_1 to v_ℓ from left to right, and have weights $w(v)$ where $v \in V(G)$, the vertex set. The integers a_i are taken from the Hirzebruch-Jung continued fraction expansion $p/q = [a_1, \dots, a_\ell]$ where $a_i \geq 2$ for $i \geq 2$. Notice that G determines a 4-manifold W' by plumbing, and that $\partial W' = S^3_{-p/q}(U)$.

3.1. A Plumbing Diagram for $S^3_{-p/q}(U)$. Consider the linear graph G in Figure 2, where $p/q = [a_1, \dots, a_\ell]$ is written in Hirzebruch-Jung continued fraction notation and $a_i \geq 2$ for $i \geq 2$ (recall that such an expansion always exists). Then G determines a sharp, simply connected, negative-definite smooth 4-manifold W' with free $H_2(W')$ by plumbing (see [7]). Moreover, $\partial W' = S^3_{-p/q}(U)$.

Following [7], it is possible to use this diagram to enumerate $\text{Spin}^c(S^3_{-p/q}(U))$ and compute the corresponding correction terms. Indeed, we observe that $H_1(W')$ is generated by $[v]$, where v is a vertex of G , and $H^2(W', \partial W')$ by $\text{PD}[v]$. Pushing this through the short exact sequence

$$0 \longrightarrow H^2(W', \partial W') \xrightarrow{Q} H^2(W') \xrightarrow{\alpha} H^2(\partial W') \longrightarrow 0,$$

we see that $\ker \alpha$ is generated by the images of the $\text{PD}[v]$ in $H^2(W')$ (i.e. by the rows of Q). We will think of $H^2(W')$ as $\text{Hom}(H_2(W'), \mathbb{Z})$, since W' is simply connected.

Now, consider all the characteristic covectors $\text{Char}(G) \subset H^2(W')$. That is, those K which satisfy

$$\langle K, [v] \rangle \equiv w(v) \pmod{2} \text{ for all } v \in V(G).$$

Since $c_1 : \text{Spin}^c(W') \rightarrow H^2(W')$ is injective with image $\text{Char}(G)$, we shall think of these covectors as the Spin^c -structures on W' . If $K \in \text{Char}(G)$, so $K = c_1(\mathfrak{s})$ for some $\mathfrak{s} \in \text{Spin}^c(W')$, then we shall write $[K]$ for the Spin^c -structure $\mathfrak{s}|_{\partial W'}$, and thus partition $\text{Char}(G)$ into equivalence classes using the relation $K \sim K'$ if $[K] \sim [K']$ (where $K, K' \in \text{Char}(G)$). Because W' is sharp (see [7]), every $\mathfrak{t} \in \text{Spin}^c(\partial W')$ lifts to some $\mathfrak{s} \in \text{Spin}^c(W')$, and hence we can think of any complete set of representatives of equivalence classes of \sim as being the Spin^c -structures on $\partial W'$.

As we would ideally like to list such representatives, it is fortunate then that the results of [7] tell us exactly how to do this. In that paper, Ozsváth and Szabó prove that $\ker U \subset HF^+(\partial W')$ is given by some subset of those K satisfying

$$w(v) + 2 \leq \langle K, v \rangle \leq -w(v) \text{ for all } v \in V(G). \quad (4)$$

To determine which such K , we start with some K satisfying (4) and let $K_0 := K$. If $\langle K_i, v \rangle = -w(v)$ for some v , we set $K_{i+1} := K_i + 2\text{PD}[v]$. Notice that $K' \sim K$. This operation is called *pushing down* (the co-ordinate of) K_i at v . Continuing like this, we conclude either with some $L := K_m$ such that

$$w(v) \leq \langle L, v \rangle \leq -w(v) - 2 \text{ for all } v \in V(G),$$

or else with an L such that there exists a v satisfying $\langle L, v \rangle \geq -w(v)$. If we conclude in the first way, we say that K initiates a *maximising* path. If we conclude in the second, we say that K initiates a *non-maximising* path. The relevant result from [7] is that $\ker U$ is given by those K that satisfy (4) and initiate maximising paths. As a corollary from

the same paper, their correction terms are computed using the formula

$$d(\partial W', \mathfrak{t}) = \max_{K: [K] = \mathfrak{t}} \frac{KQ^{-1}K^t + |G|}{4}. \quad (5)$$

We remind the reader that as $\partial W'$ an L -space, $\ker U$ gives us the complete collection of Spin^c -structures on $\partial W'$ without repetition.

3.2. Correction Term Calculus. At this point, it is natural to ask what the Spin^c -structures on $S^3_{-p/q}(U)$ look like after applying the above algorithm. To answer this question, we require some definitions.

Definition 3.1. Suppose that Y is a closed 3-manifold contained in X , a smooth 4-manifold. Then given some $\mathfrak{s} \in \text{Spin}^c(X)$, we say that $c_1(\mathfrak{s})$ is a maximiser for $\mathfrak{t} = \mathfrak{s}|_Y$ if $c_1(\mathfrak{s})^2$ is maximal among $c_1(\mathfrak{s}')^2$, where $\mathfrak{s}' \in \text{Spin}^c(X)$ satisfies $\mathfrak{s}'|_Y = \mathfrak{t}$.

Definition 3.2. Let K be a characteristic covector for the linear graph G in Figure 2 which satisfies

$$w(v) \leq \langle K, v \rangle \leq -w(v) \text{ for all } v \in V(G).$$

Then we say that a vertex v_i is a peak for K if $\langle K, [v_i] \rangle = a_i$, and call $K + 2\text{PD}[v_i]$ the push-down of K at v_i (we also call any covector obtained by a sequence of such moves a push-down of K). We say that K contains no full tanks if there do not exist $i < j$ such that v_i and v_j are peaks and $\langle K, [v_k] \rangle = a_k - 2$ for all $i < k < j$. We say that K is left-full if there exists a peak v_i such that $\langle K, [v_k] \rangle = a_k - 2$ for all $k < i$. To make notation simpler we will write $b_k := 2 - a_k$.

Lemma 3.3. The Spin^c -structures on $S^3_{-p/q}(U)$ are represented by those characteristic covectors K satisfying (4) that contain no full tanks. We call the collection of such characteristic covectors \mathcal{K} .

Proof. We first prove that if K has a full tank then it initiates a non-maximising path. Indeed, observe the following path (presenting only the relevant section of K):

$$\begin{aligned} (a_i, -b_{i+1}, -b_{i+2}, \dots, -b_{j-1}, a_j) &\longrightarrow (-a_i, a_{i+1}, -b_{i+2}, \dots, -b_{j-1}, a_j) \\ &\longrightarrow (b_i, -a_{i+1}, a_{i+2}, \dots, -b_{j-1}, a_j) \\ &\longrightarrow (b_i, b_{i+1}, -a_{i+2}, \dots, -b_{j-1}, a_j) \\ &\longrightarrow \dots \\ &\longrightarrow (b_i, b_{i+1}, b_{i+2}, \dots, a_{j-1}, a_j) \\ &\longrightarrow (b_i, b_{i+1}, b_{i+2}, \dots, -a_{j-1}, a_j + 2). \end{aligned}$$

Here $\langle L, [v_j] \rangle > -w(v_j)$, so the initiated path is non-maximising.

What remains to be shown is that if K does not have a full tank, then it initiates a maximising path. We do this by inducting on the number peaks in K and its push-downs. If there are none, we have a (trivial) maximising path. Thus, we presume there is at least one peak at v_i . Now, push down at v_i . Depending on whether $\langle K, [v_{i\pm 1}] \rangle = a_{i\pm 1} - 2$, there are three possibilities for the new $K' = K + 2\text{PD}[v_i]$:

- (1) v_{i-1} and v_{i+1} are not peaks of K' . Then K' has one peak fewer than K and also contains no full tanks as $\langle K', [v_i] \rangle = -a_i \neq a_i - 2$ since $a_i \geq 2$ when $i \geq 2$. Hence, we apply the inductive hypothesis.
- (2) v_{i-1} is not a peak, v_{i+1} is (or the reverse situation). In this case, push down at v_{i+1} , and continue pushing down at any further peaks this generates, necessarily heading to the right. As K had no full tanks, this process must stop without

initiating a non-maximising path. As in the previous case, the resulting covector has one peak fewer than K and no full tanks, so apply the induction hypothesis.

- (3) v_{i-1} and v_{i+1} are peaks. This situation is the same as the one above, pushing down in both directions unilaterally until the process halts. If $a_i = 2$, we will have to repeat this whole procedure multiple times, but eventually it will halt.

In all situations we have a maximising path. This completes our proof. \square

Since $S^3_{-p/q}(U)$ is an L -space, we have now isolated a collection $\mathcal{K} \subset \text{Char}(G)$ in bijection with $\text{Spin}^c(S^3_{-p/q}(U))$. Hence, we have the following proposition (the last part of which is an application of (5)).

Proposition 3.4. *Given $\mathfrak{t} \in \text{Spin}^c(S^3_{-p/q}(U))$, there is a unique $K \in \mathcal{K}$ such that $[K] = \mathfrak{t}$ and K is a maximiser for $[K]$. Moreover,*

$$d(S^3_{-p/q}(U), \mathfrak{t}) = \frac{KQ^{-1}K^t + b_2(W')}{4}.$$

3.3. Comparing the Rational and Integral Surgeries. We are now ready to make comparisons between the $-p/q$ and $-n$ surgeries on C and U . In doing so, it is very important to keep track of which coefficients and which knots we are considering. Thus, we observe the following:

- (1) Let $W(C) : S^3 \rightarrow S^3_{-p/q}(C)$ be the cobordism determined by the diagram in Figure 1, and let $W := W(C) \cup_{S^3} D^4$ and $W' := W(U) \cup_{S^3} D^4$ (i.e. these manifolds are the traces of the surgeries in Figure 1). Note that this W' and the W' of the previous section are identical;
- (2) The intersection form of the cobordism $W(C)$ is independent of C . Ergo, W and W' have the same intersection form, represented in some bases by the adjacency matrix Q of the graph G in Figure 2;
- (3) Courtesy of this fact, a $K \in \text{Char}(G)$ is a maximiser for $[K] \in \text{Spin}^c(S^3_{-p/q}(U))$ if and only if K is also a maximiser for the corresponding Spin^c -structure on $S^3_{-p/q}(C)$, which we shall also denote by $[K]$. The crucial difference is that K might not compute the correction term for $S^3_{-p/q}(C)$. Henceforth, we shall think of $\text{Char}(G)$ as the Spin^c -structures on either 4-manifold W or W' , and any complete collection \mathcal{F} of representatives of equivalence classes of \sim as the Spin^c -structures on either 3-manifold $S^3_{-p/q}(C)$ or $S^3_{-p/q}(U)$;
- (4) $W(C)$ splits naturally into two cobordisms,

$$S^3 \xrightarrow{W_1(C)} S^3_{-n}(C) \xrightarrow{W_2(C)} S^3_{-p/q}(C).$$

All three cobordisms are negative definite and have intersection forms independent of C .

Now consider $W_2(C)$. Given a maximiser K which determines $\mathfrak{t} \in \text{Spin}^c(S^3_{-p/q}(C))$, this K also determines a $\mathfrak{t}' \in \text{Spin}^c(S^3_{-n}(C))$ by considering the value of $k := \langle K, [v_1] \rangle$ modulo $2n$. Comparing this with the labelling used in Proposition 2.2, we see that \mathfrak{t}' corresponds with $\mathfrak{t}_{\frac{k+n}{2}} \in \text{Spin}^c(S^3_{-n}(C))$, unless $k = n$, in which case \mathfrak{t}' corresponds with $\mathfrak{t}_0 \in \text{Spin}^c(S^3_{-n}(C))$.

Lemma 3.5. *Let $K \in \mathcal{K}$. Then,*

- (1) *If $\langle K, [v_1] \rangle \neq n$, then there is a q -to-one map $\mathcal{K} \rightarrow \text{Char}(v_1)$ given by restriction to the first co-ordinate;*

- (2) If $\langle K, [v_1] \rangle = n$, then there are $q - r$ elements of \mathcal{K} which map to $(n) \in \text{Char}(v_1)$, also by restriction.

Proof. Consider the case when $\langle K, [v_1] \rangle \neq n$. Then the number of covectors K that restrict to $\langle K, [v_1] \rangle$ is equal to the number of characteristic covectors on the linear graph $G - v_1$ which satisfy (4), initiate maximising paths, and have no full tanks. This is just the number of Spin^c -structures on the lens space given by $[a_2, \dots, a_\ell]$ -surgery on the unknot. As $[a_2, \dots, a_\ell] = q/r$, we are done.

For the second claim, observe that there are $n - 1$ values of $\langle K, [v_1] \rangle$ covered by the above case, together accounting for $(n - 1)q$ of the elements of \mathcal{K} . Therefore, the remaining $q - r$ must restrict to $(n) \in \text{Char}(v_1)$. \square

3.4. Adjusting the Maximisers. It is proved in [6] that if \overline{W} is a negative-definite cobordism from Y_1 to Y_2 , both rational homology spheres, then

$$\left\{ \frac{c_1(\overline{\mathfrak{s}})^2 + b_2(\overline{W})}{4} \right\} \leq d(Y_2, \overline{\mathfrak{s}}|_{Y_2}) - d(Y_1, \overline{\mathfrak{s}}|_{Y_1}) \quad (6)$$

for any $\overline{\mathfrak{s}} \in \text{Spin}^c(\overline{W})$. Applying this to $\overline{W} = W_2(U)$, we observe that if $\overline{\mathfrak{s}}$ is the restriction of some $\mathfrak{s} \in \text{Spin}^c(W')$ satisfying $c_1(\mathfrak{s}) \in \mathcal{K}$, then we have equality in (6) (c.f. the proof of Lemma 4.3 in [8] and Proposition 3.4). This allows us to compute the term in curly braces and, since the intersection form of $W_2(C)$ is independent of C , substitute it into (6) for $\overline{W} = W_2(C)$. Consequently, if \mathfrak{t}' and \mathfrak{t} are cobordant via an element of \mathcal{K} ,

$$D_C^n(\mathfrak{t}') \leq D_C^{p/q}(\mathfrak{t}). \quad (7)$$

A very similar argument, applying (6) to $W_1(C)$, tells us that

$$0 \leq D_C^n(\mathfrak{t}'). \quad (8)$$

This argument on \mathcal{K} is just a special case of the following lemma.

Lemma 3.6. *Suppose that $\mathcal{F} \subset \text{Char}(G)$ is a complete set of representatives of equivalence classes of \sim , and that every $K_i \in \mathcal{F}$ is a maximiser for $[K_i]$. Then on defining \mathfrak{s}_i by $c_1(\mathfrak{s}_i) = K_i$ and setting $w_i = \mathfrak{s}_i|_{S_{-p/q}^3(C)}$ and $v_i = \mathfrak{s}_i|_{S_{-n}^3(C)}$, it follows that*

$$0 \leq D_C^n(v_i) \leq D_C^{p/q}(w_i).$$

Proof. This is a combination of (8) on the left and (7) on the right. \square

As it turns out, \mathcal{K} is not the optimal choice of representatives for our purposes, since it does not yield a function \mathfrak{r} satisfying Theorem 1.1. We therefore ask: If K is a maximiser, are there any other $K' \sim K$ that are maximisers? The answer is yes.

Lemma 3.7. *Let $\langle K, [v_i] \rangle = a_i$, where $K \in \text{Char}(G)$ (not necessarily in \mathcal{K}). Then $K' := K + 2\text{PD}[v_i]$ satisfies $(K')^2 = K^2$.*

Proof. Recall that $\text{PD}[v_i]$, viewed as an element of $H^2(W)$, is the i -th row of Q . Hence, $\text{PD}[v_i]Q^{-1} = e_i$, the i -th standard basis vector. Thus,

$$\begin{aligned} (K + 2\text{PD}[v_i])^2 &= (K + 2\text{PD}[v_i])Q^{-1}(K + 2\text{PD}[v_i])^t \\ &= KQ^{-1}K^t + 4\text{PD}[v_i]Q^{-1}K^t + 4\text{PD}[v_i]Q^{-1}\text{PD}[v_i]^t \\ &= KQ^{-1}K + 4e_iK^t + 4e_i\text{PD}[v_i]^t \\ &= KQ^{-1}K + 4\langle K, [v_i] \rangle - 4a_i, \end{aligned}$$

and as $\langle K, [v_i] \rangle = a_i$, we are done. \square

Corollary 3.8. *If K' is a push-down of $K \in \mathcal{K}$, then K' is a maximiser of $[K]$.*

Corollary 3.9. *Let \mathcal{M} be the set of all maximisers in $\text{Char}(G)$. Then if $K \in \mathcal{M}$, so are all its push-downs.*

Finally, we are able to prove the following critical lemma of this section.

Lemma 3.10. *Let C be a knot in S^3 , then*

$$\sum_{\mathfrak{t} \in \text{Spin}^c(S^3_{-p/q}(C))} D_C^{p/q}(\mathfrak{t}) \geq q \cdot \sum_{\mathfrak{t} \in \text{Spin}^c(S^3_{-n}(C))} D_C^n(\mathfrak{t}) - r \cdot \min_{\mathfrak{t} \in \text{Spin}^c(S^3_{-n}(C))} \{D_C^n(\mathfrak{t})\}.$$

Proof. Construct a family \mathcal{K}' of characteristic covectors for use in Lemma 3.6 as follows. If $K \in \mathcal{K}$ satisfies

- (1) $\langle K, [v_1] \rangle = j$ for some $-1 \leq j < n$; and
- (2) $K|_{G-v_1}$ is left full,

then let $K' := K + 2 \sum_{i=1}^k \text{PD}[v_i]$ be a member of \mathcal{K}' . Here $k \geq 2$ is the smallest integer such that v_k is a peak for K (guaranteed to exist by the second condition above). This clearly determines the same Spin^c -structure on the boundary manifolds, and is a maximiser by Corollary 3.8.

For all other $K \in \mathcal{K}$, let K be a member of \mathcal{K}' . The family \mathcal{K}' is now clearly a complete set of representatives for the equivalence classes of \sim , each element of which is a maximiser. We claim that the desired result is obtained by adding up all inequalities in Lemma 3.6, using $\mathcal{F} = \mathcal{K}'$.

To prove this claim, let us consider what the pushing down does. Our first piece of information is that $\langle K', [v_1] \rangle = j + 2$, so we are “nudging K up” the values in the first co-ordinate. We claim that, for a given j , we have nudged up precisely r different K . Indeed, recall from Lemma 3.5 that there are $q - r$ elements of \mathcal{K} with $\langle K, [v_1] \rangle = n$. Another way of computing this number is:

$$\#\mathcal{K}|_{G-v_1} - \# \left\{ \begin{array}{c} \text{left-full elements of} \\ \mathcal{K}|_{G-v_1} \end{array} \right\}.$$

Since the first term here is q (as a scholium of Lemma 3.5), the second term must be r , as required.

This calculation completed, we now observe that \mathcal{K}' has q elements that restrict to $(j) \in \text{Char}(v_1)$ for any $-n + 2 \leq j \leq n$, except $j = -1$ if n is odd or $j = 0$ if n is even, when there are $q - r$ such elements. Our lemma follows by applying Lemma 3.6 and adding up all the inequalities. Notice that the exceptional Spin^c -structure is one with minimal deficiency (see Lemma 2.4). \square

4. PROOFS OF THE THEOREMS

Now that all the machinery is in place, we can rapidly prove Theorem 1.1.

Proof of Theorem 1.1. Take Lemma 3.10, observing that we actually have equality by Lemma 2.3. This implies that all the right hand inequalities in Lemma 3.6 were in fact equalities induced by the members of \mathcal{K}' . The result follows. \square

As remarked after the proof of Lemma 2.3, we had actually already proved Theorem 1.1 some time ago. However, this more recent proof has the advantage that it gives us insights the previous one did not: it allows us to see how \mathfrak{r} behaves. Indeed, take any $\mathfrak{t} \in \text{Spin}^c(S^3_{-p/q}(C))$ and some maximiser K for \mathfrak{t} . Then the value $\mathfrak{r}(\mathfrak{t})$ is determined by finding the $K' \in \mathcal{K}'$ such that $K \sim K'$; $\mathfrak{r}(\mathfrak{t})$ is the Spin^c -structure on $S^3_{-n}(C)$ determined by the maximiser $(\langle K', [v_1] \rangle)$.

Corollary 4.1. *With notation as above, $D^{p/q}(\mathbf{t})$ is minimal if $\langle K', [v_1] \rangle = 0, \pm 1$. If, additionally, n is even and there are $q - r + 1$ choices of \mathbf{t} such that $D^{p/q}(\mathbf{t})$ is minimal, then this extends to $\langle K', [v_1] \rangle = \pm 2$.*

In either case, if $D^{p/q}(\mathbf{t}) = 0$ for some \mathbf{t} , then the minimal deficiency is zero.

Proof. The first statement is an immediate consequence of Theorem 1.1 and Lemma 2.4. The second arises because there are only $q - r$ such \mathbf{t} with first co-ordinate 0, and because those K' with $\langle K', [v_1] \rangle = \pm 2$ have the next smallest deficiencies (by an argument almost identical to that in Lemma 2.4). The final comment is a trivial by-product of Lemma 3.6 as the deficiencies are non-negative. \square

As mentioned in Section 1, this knowledge of \mathbf{r} allows us to turn Theorem 1.1 into an obstruction, given Y , to $Y = S^3_{-p/q}(C)$ (under certain extra circumstances). We have already stated this obstruction as Theorem 1.2, but to prove it we must establish some algebraic preliminaries. For greater detail on these preliminaries, we refer the reader to Lemma 2.3 of [3] and Section 3.2 of [2]. We have summarised the key results below.

Proposition 4.2. *Let Y be a 3-manifold obtained by integral surgery on an ℓ -component link L with negative-definite linking matrix Q and trace W ; let also Y' be a manifold obtained by integral surgery on a (possibly different) link L' , also with linking matrix Q , but whose trace W' is sharp. Finally, suppose that $\partial X = -Y$, where X is a sharp, negative-definite smooth 4-manifold, that $H_1(X) = 0$, and that $H_2(X)$ is free. Then since c_1 commutes with the restriction maps on $\text{Spin}^c(\cdot)$ and $H^2(\cdot)$ induced by inclusion of a 3- or 4-manifold into a 4-manifold, there is a bijection*

$$\begin{aligned} \{\mathfrak{s} \in \text{Spin}^c(X \cup_Y W) \mid \mathfrak{s}|_Y = \mathbf{t}\} &\longrightarrow \left\{ (\mathfrak{s}_X, \mathfrak{s}_W) \in \text{Spin}^c(X) \times \text{Spin}^c(W) \mid \begin{array}{l} \mathfrak{s}_X|_Y = \mathbf{t} \\ \mathfrak{s}_W|_Y = \mathbf{t} \end{array} \right\} \\ \mathfrak{s} &\longmapsto (\mathfrak{s}|_X, \mathfrak{s}|_W) \end{aligned} \quad (9)$$

such that

$$c_1(\mathfrak{s})^2 = c_1(\mathfrak{s}|_X)^2 + c_1(\mathfrak{s}|_W)^2.$$

Moreover, given $\mathbf{t} \in \text{Spin}^c(Y)$, there is some $\mathfrak{s} \in \text{Spin}^c(X \cup_Y W)$ such that

$$\max_{\substack{\mathfrak{s} \in \text{Spin}^c(X \cup_Y W) \\ \mathfrak{s}|_Y = \mathbf{t}}} c_1(\mathfrak{s})^2 + b_2(X \cup_Y W) = 4d(Y', \mathbf{t}) - 4d(Y, \mathbf{t}). \quad (10)$$

Note that in [2], the additional assumption was made that $\det(Q)$ is odd. This, however, is not necessary: its only function was to ensure that $\text{Spin}^c(X) \rightarrow \text{Spin}^c(Y)$ and $\text{Spin}^c(W) \rightarrow \text{Spin}^c(Y)$ surject. This is assured by the fact that $H_1(X)$ and $H_1(W)$ are torsion-free (c.f. [3]).

Taking this proposition as given, it follows that a maximiser $c_1(\mathfrak{s})$ decomposes into a pair of maximisers $(c_1(\mathfrak{s}|_X), c_1(\mathfrak{s}|_W))$. To see what this decomposition looks like, at least on W , we use the diagram

$$H_2(X) \oplus H_2(W) \longrightarrow H_2(X \cup_Y W) \simeq H^2(X \cup_Y W) \longrightarrow H^2(X) \oplus H^2(W). \quad (11)$$

If we employ a basis $\{u_1, \dots, u_\ell\}$ for $H_2(W)$ with images $\{\overline{u_1}, \dots, \overline{u_\ell}\}$ in $H_2(X \cup_Y W)$, a class $\alpha \in H^2(W \cup_Y X)$ restricts to the class $(\langle \alpha, \overline{u_1} \rangle, \dots, \langle \alpha, \overline{u_\ell} \rangle) \in H^2(W)$ when written in the dual basis $\{u_1^*, \dots, u_\ell^*\}$. In particular this applies when $\alpha = c_1(\mathfrak{s})$: the restriction $c_1(\mathfrak{s}|_W)$ has the form $(\langle c_1(\mathfrak{s}), \overline{u_1} \rangle, \dots, \langle c_1(\mathfrak{s}), \overline{u_\ell} \rangle)$.

Now suppose that K is a maximiser for $[K]$ and that $K = c_1(\mathfrak{s}_W)$ for some $\mathfrak{s}_W \in \text{Spin}^c(W)$. Suppose also that if we put $\mathbf{t} = [K]$ in (10), the RHS vanishes. Then since X is sharp, there is some $\mathfrak{s}' \in \text{Spin}^c(X \cup_Y W)$ with $\alpha' = c_1(\mathfrak{s}')$ such that

$$-b_2(X \cup_Y W) = (\alpha')^2 = c_1(\mathfrak{s}'|_X)^2 + c_1(\mathfrak{s}'|_W)^2.$$

As we know that $c_1(\mathfrak{s}'|_W)$ is also a maximiser for \mathfrak{t} , it follows that its square is K^2 . Thus, letting $\mathfrak{s} \in \text{Spin}^c(X \cup_Y W)$ correspond to $(\mathfrak{s}'|_X, \mathfrak{s}_W)$ under (9), and putting $\alpha = c_1(\mathfrak{s})$, we have

$$(\alpha')^2 = c_1(\mathfrak{s}'|_X)^2 + c_1(\mathfrak{s}'|_W)^2 = c_1(\mathfrak{s}'|_X)^2 + K^2 = c_1(\mathfrak{s}'|_X)^2 + c_1(\mathfrak{s}_W)^2 = \alpha^2.$$

Hence $\alpha^2 + b_2(X \cup_Y W) = 0$.

Since $X \cup_Y W$ is a closed, simply connected, negative-definite smooth 4-manifold, it follows from Donaldson's diagonalisation theorem [1] that

$$(H^2(X \cup_Y W), Q_{X \cup_Y W}) \simeq (\mathbb{Z}^{b_2(X) + b_2(W)}, -\text{id})$$

as lattices. Thus, since α is a characteristic covector of $Q_{X \cup_Y W}$, it follows that $\alpha \equiv (1, 1, \dots, 1) \pmod{2}$, whence all entries of α are ± 1 . Summarised, we have the following lemma.

Lemma 4.3. *Let K be a maximiser for $\mathfrak{t} = [K]$ such that*

$$d(Y', \mathfrak{t}) - d(Y, \mathfrak{t}) = 0.$$

Then there is some $\alpha \in \{\pm 1\}^{b_2(X) + \ell}$ such that $K = (\langle \alpha, \overline{u}_1 \rangle, \dots, \langle \alpha, \overline{u}_\ell \rangle)$, written in the dual basis $\{u_1^, \dots, u_\ell^*\}$.*

To apply this lemma, we let $Y' = S_{-p/q}^3(U)$ (whose trace W' is sharp), and $Y = S_{-p/q}^3(C)$ (with trace W). Theorem 1.2 is now finally within reach.

4.1. Proof of Theorem 1.2 when $p > q > 1$. Let our basis $\{u_1, \dots, u_\ell\}$ above be given by the vertices $[v_i]$ of our graph G , and consider the diagram below:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^1(Y) & \longrightarrow & H^2(X, \partial X) \oplus H^2(W, \partial W) & \longrightarrow & H^2(X \cup_Y W) \longrightarrow \dots \\ & & \text{PD} \downarrow & & \text{PD} \downarrow & & \text{PD} \downarrow \\ \dots & \longrightarrow & H_2(Y) & \longrightarrow & H_2(X) \oplus H_2(W) & \xrightarrow{A} & H_2(X \cup_Y W) \longrightarrow \dots \end{array}$$

Here, the two groups on the left must vanish because the vertical maps are isomorphisms and $H^1(Y) = 0$. Hence, the lattice underlying $Q_X \oplus Q_W$ embeds in the one underlying $Q_{X \cup_Y W}$, and on passing to the lower row, there must exist some matrix A with integral entries such that $-AA^t = Q_X \oplus Q_W$. When expressed like this, the last ℓ rows of A are the images of the $[v_i]$ in $H_2(X \cup_Y W)$, and we shall use Lemma 4.3 to prove our claims about their structure. It is helpful to keep in mind that the (i, j) -th entry of $Q_X \oplus Q_W$ is in fact the standard negative-definite inner product of the i -th and j -th rows of A . We label the last ℓ rows of A by x, y_2, \dots, y_ℓ .

Our first task is to establish the structure of y_i for $i = 2, \dots, \ell$. This has three parts: first, we show that all the non-zero entries are unital; second, that non-adjacent rows have no non-zero entries in the same spots; and third, that adjacent rows share only one spot with non-zero entries and that these overlapping entries are opposite in sign.

To achieve the first of these objectives, consider $y_i = (y_{i,j})_j$ for $2 \leq i \leq \ell$. Recall that $b_j := 2 - a_j$ and define $K \in \text{Char}(G)$ by

$$K = \begin{cases} (0, b_2, \dots, b_{i-1}, a_i, b_{i+1}, \dots, b_\ell) & \text{if } n \text{ is even} \\ (-1, b_2, \dots, b_{i-1}, a_i, b_{i+1}, \dots, b_\ell) & \text{if } n \text{ is odd} \end{cases}.$$

Since there are no full tanks in K it is clear that $K \in \mathcal{K}$ and hence it is a maximiser. To determine the value of $D^{p/q}([K])$, we need to find the first co-ordinate of the corresponding $K' \in \mathcal{K}'$ such that $K' \sim K$ (by Theorem 1.1). If there is some $a_j \neq 2$ for $2 \leq j < i$

then $K \in \mathcal{K}'$. Otherwise, consider $K' = K + 2 \sum_{j=2}^i \text{PD}[v_j] \in \mathcal{K}'$, which has first co-ordinate 2 or 1 depending on parity. By our hypothesis on the deficiencies, it follows that $D^{p/q}([K]) = 0$ (via Corollary 4.1). Thus by Lemma 4.3, there is an $\alpha \in \{\pm 1\}^{b_2(X)+\ell}$ such that $\langle \alpha, y_i \rangle = a_i$. Rephrased,

$$-\sum_j \alpha_j y_{i,j} = a_i = \sum_j y_{i,j}^2,$$

where the right hand side comes from the fact that $y_i^2 = a_i$. Consequently,

$$\sum_j (y_{i,j}^2 + \alpha_j y_{i,j}) = 0,$$

and since $\alpha_j = \pm 1$, each summand is non-negative. Therefore, each summand must vanish, which in turn requires $y_{i,j} \in \{-1, 0, 1\}$. It is clear that for exactly a_i values of j , $y_{i,j} \neq 0$.

With this step done, we now need to establish how the rows line up with each other. Thus, consider $2 \leq i < j \leq \ell$ such that $j - i \geq 2$, set $m := a_i$, and permute the basis of $H_2(X \cup_Y W)$, changing signs as necessary, so that $y_i = (1, \dots, 1, 0, \dots, 0)$. As before, there must be an $\alpha = c_1(\mathfrak{s})$ such that $c_1(\mathfrak{s}|_W) = K$ where

$$K = \begin{cases} (0, b_2, \dots, b_{i-1}, a_i, -a_{i+1}, b_{i+2}, \dots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \dots, b_\ell) & \text{if } n \text{ is even} \\ (-1, b_2, \dots, b_{i-1}, a_i, -a_{i+1}, b_{i+2}, \dots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \dots, b_\ell) & \text{if } n \text{ is odd} \end{cases},$$

which is obtained by pushing down at v_{j-1} in

$$(*, b_2, \dots, b_{i-1}, a_i - 2, -b_{i+1}, -b_{i+2}, \dots, -b_{j-2}, a_{j-1}, a_j - 2, b_{j+1}, \dots, b_\ell) \in \mathcal{K}$$

and repeating to the left. In either case, exactly as before we find that $D^{p/q}([K]) = 0$ by showing the first co-ordinate of the corresponding K' is one of $0, \pm 1, 2$. Then there is an α such that $\langle \alpha, y_i \rangle = a_i$ and $\langle \alpha, y_j \rangle = a_j$. The first of these statements tells us that $\alpha_k = -1$ for all $k = 1, \dots, m$.

Now, let $I = \{k \leq m | y_{j,k} \neq 0\}$. We claim that $I = \emptyset$. Indeed, as

$$-a_j = -\langle \alpha, y_j \rangle = \sum_{k \in I} \alpha_k y_{j,k} + \sum_{k > m} \alpha_k y_{j,k},$$

each summand on the RHS must be -1 or 0 . We know that $\alpha_k = -1$ for $k \in I$, so $y_{j,k} = 1$ for $k \in I$. Yet $y_i \cdot y_j = 0$, so $\sum_{k \in I} 1 = 0$, and $I = \emptyset$.

We repeat a similar argument for $j = i + 1$, though our goal is to show that there is a unique element $k \in I$ and that $y_{i+1,k} = -1$. For $2 \leq i \leq \ell$ we take

$$K = \begin{cases} (0, b_2, \dots, b_{i-1}, a_i, a_{i+1} - 2, b_{i+2}, \dots, b_\ell) & \text{if } n \text{ is even} \\ (-1, b_2, \dots, b_{i-1}, a_i, a_{i+1} - 2, b_{i+2}, \dots, b_\ell) & \text{if } n \text{ is odd} \end{cases},$$

and note again that $D^{p/q}([K]) = 0$. Permuting and changing signs as necessary, we may assume that $y_i = (1, \dots, 1, 0, \dots, 0)$ and define I as before. This time, however,

$$-a_{i+1} + 2 = -\langle \alpha, y_{i+1} \rangle = \sum_{k \in I} \alpha_k y_{i+1,k} + \sum_{k > m} \alpha_k y_{i+1,k},$$

and exactly one summand on the RHS is 1 . If that summand is in the second sum then all summands in the first are negative, so $y_{i+1,k} = 1$ for all $k \in I$. But then $-1 = y_i \cdot y_{i+1} = \sum_{k \in I} 1$, a contradiction. Therefore $y_{i+1,k} = -1$ for precisely one $k \in I$, and by an argument similar to the one just made, this k is the unique element of I .

At this point, up to permuting the basis of $H_2(X \cup_Y W)$ and changing signs as necessary, we have established the form of the last $\ell - 1$ rows of A . What remains is to establish x .

With this in mind, our first goal is to prove that it has the shape $(*, \dots, *, 1, 0, \dots, 0)$ as outlined in the statement of the theorem.

Fix $i \in \{3, \dots, \ell\}$, let x_1, \dots, x_m be the entries of x in the same spots as the non-zero entries of row y_i , and let $x_{m+1}, \dots, x_{b_2(X)+\ell}$ be the rest. Note that $m = a_i$. Again, change signs as necessary so that $y_{i,k} \geq 0$ for all k . Then as $x \cdot y_i = 0$, it follows that

$$x_1 + \dots + x_{m-1} + x_m = 0. \quad (12)$$

Our goal is to show that $x_k = 0$ for all $k \leq m$.

Define a set

$$\mathcal{S} = \{K \in \mathcal{M} \mid \langle K, [v_i] \rangle = m, D^{p/q}([K]) = 0\}.$$

Then for any $K \in \mathcal{S}$, we find an α according to Lemma 4.3 such that $\langle \alpha, y_i \rangle = a_i = m$. Hence, $\alpha_1 + \dots + \alpha_m = -m$, whence $\alpha_k = -1$ for $k \leq m$. Indeed, all $\alpha \in \{\pm 1\}^{b_2(X)+\ell}$ satisfying these equations determine some $K \in \mathcal{S}$. Let $j = \langle K, [v_1] \rangle = \langle \alpha, x \rangle$. Then

$$j = x_1 + \dots + x_m - \sum_{k>m} \alpha_k x_k = - \sum_{k>m} \alpha_k x_k,$$

where we used (12) to obtain the last equality. Thus the maximum value for j as we vary $K \in \mathcal{S}$ is

$$\sum_{k>m} |x_k|. \quad (13)$$

Now suppose, without loss of generality, that $x_m \leq x_k$ for all $k < m$, and define

$$\mathcal{S}' = \{K \in \mathcal{M} \mid \langle K, [v_i] \rangle = m - 2, D^{p/q}([K]) = 0\}.$$

Then similarly there is some $\beta \in \{\pm 1\}^{b_2(X)+\ell}$ such that $\langle \beta, y_i \rangle = m - 2$, whence $\beta_k = -1$ for all values of $k \leq m$ except one. As before, all such β determine a $K \in \mathcal{S}'$. Hence

$$j = - \sum_{k=1}^m \beta_k x_k - \sum_{k>m} \beta_k x_k \quad (14)$$

attains its maximal value when $\beta_k = -1$ for all $k < m$ and $\beta_m = 1$ (by choice of x_m). This maximal value is

$$\sum_{k<m} x_k - x_m + \sum_{k>m} |x_k|. \quad (15)$$

We claim that the two maxima given by (13) and (15) are in fact identical. Indeed, let $j_{\max} \leq n$ be the maximal integer j such that $D^n([j]) = 0$ (note that $j_{\max} \geq 1$ by assumption on the number of vanishing deficiencies). Then if j_{\max} can be attained by elements of \mathcal{S} and \mathcal{S}' , the claim must be true (since larger values are ruled out by the deficiency condition). Observe that

$$K = (j_{\max}, -a_2, b_3, \dots, b_{i-2}, b_{i-1}, m, b_{i+1}, \dots, b_\ell)$$

satisfies $D^{p/q}([K]) = 0$, since $K \in \mathcal{K}'$: it is obtained by pushing down

$$(j_{\max} - 2, -b_2, -b_3, \dots, -b_{i-2}, a_{i-1}, m - 2, b_{i+1}, \dots, b_\ell) \in \mathcal{K}$$

at v_{i-1} and to the left. Similarly,

$$K' = (j_{\max}, b_2, \dots, b_{i-1}, m - 2, b_{i+1}, \dots, b_\ell) \in \mathcal{S}' \cap \mathcal{K}'.$$

Thus the maximal values of j are the same in both families. Hence,

$$\sum_{k<m} x_k - x_m = 0.$$

However, using (12) to rewrite the first term, we find that $x_m = 0$. Therefore $x_k \geq 0$ for all $k \leq m$, by choice of x_m , and from (12) again we find that $x_k = 0$.

Now consider row $i = 2$ and set $m = a_2$. We wish to show that $x_k = 0$ for all $k \leq m$ except one, for which $x_k = -1$. In this case, (12) becomes

$$x_1 + \cdots + x_{m-1} + x_m = -1. \quad (16)$$

Although we will keep the set \mathcal{S}' as defined before, this time we use

$$\mathcal{S} = \{K \in \mathcal{M} \mid \langle K, [v_2] \rangle = -m, D^{p/q}([K]) = 0\},$$

so that the maximum (13) becomes

$$1 + \sum_{k > m} |x_k|.$$

while the second maximum (15) remains unchanged after we have defined x_m to be the smallest of the x_k for $k \leq m$. We claim that the two maxima are equal. Indeed, observe that $(j_{\max}, -m, b_3 + 2, b_4, \dots, b_\ell) \in \mathcal{S} \cap \mathcal{K}'$, and $(j_{\max}, m - 2, b_3, \dots, b_\ell) \in \mathcal{S}' \cap \mathcal{K}'$. Hence, comparing the maxima, we obtain:

$$\sum_{k < m} x_k - x_m = 1.$$

If we rearrange (16), as before, we find that $x_m = -1$. If $m = 2$, then (16) yields the result. If, on the other hand, $m > 2$, then repeat this process with

$$\mathcal{S}'' = \{K \in \mathcal{M} \mid \langle K, [v_2] \rangle = m - 4, D^{p/q}([K]) = 0\}$$

and x_{m-1} defined to be the next smallest after x_m . We find that $x_{m-1} + x_m = -1$, from which $x_{m-1} = 0$. Hence $x_k \geq 0$ for all $k \leq m - 1$, and it follows from (16) that $x_k = 0$ for $k \leq m - 1$, as required.

By this point we are finally almost there. What remains to establish is the changemaker condition on x . Using the labels σ_i established, and defining σ_0 to be the other unital entry, change signs as usual so that $\sigma_i \geq 0$. Let

$$J := \{\langle K, [v_1] \rangle \mid K \in \mathcal{M}, \langle K, [v_2] \rangle = -a_2, D^{p/q}([K]) = 0\},$$

and observe that J consists of all values $j \equiv n$ from $2 - j_{\max}$ to j_{\max} . The asymmetry is a result of the fact that if $K = (j, -a_2, *, \dots, *)$ is a relevant maximiser with appropriate values $*$, then $K \in \mathcal{K}'$ if $j \geq 1$, whereas $K \sim K' := (j - 2, a_2, *, \dots, *) \in \mathcal{K}'$ if $j < 1$. Thus, in light of the evaluation on $[v_2]$,

$$j = \sigma_0 - \sum_{i \geq 1} \alpha_i \sigma_i = 1 - \sum_{i \geq 1} \alpha_i \sigma_i$$

attains these values too. By writing $\alpha_i = -1 + 2\chi_i$ (where $\chi_i \in \{0, 1\}$), we obtain

$$j = 1 + \sum_{i \geq 1} \sigma_i - 2 \sum_{i \geq 1} \chi_i \sigma_i = j_{\max} - 2 \sum_{i \geq 1} \chi_i \sigma_i,$$

and thus $\{\sum_{i \geq 1} \chi_i \sigma_i \mid \chi_i \in \{0, 1\}\}$ consists of all integers from 0 to $\sum_{i \geq 1} \sigma_i$. This is precisely the condition for a changemaker set.

4.2. Proof of Theorem 1.2 when $0 < p < q$. This proof is extremely similar to the previous one, so we only outline the differences. Crucially, $n = 1$, so via Theorem 1.1 it follows that all the deficiencies $D^{p/q}([K])$ vanish for any maximiser K . This fact makes the proof much easier.

To ensure that all non-zero entries in y_i are ± 1 for all i , we use the maximiser

$$K = (1, b_2, \dots, b_{i-2}, -a_{i-1}, a_i, b_{i+1}, \dots, b_\ell).$$

We know that this choice of K is in fact a maximiser since it is a push-down of

$$(1, -b_2, \dots, -b_{i-2}, -b_{i-1}, -b_i, b_{i+1}, \dots, b_\ell) \in \mathcal{K}.$$

The same argument as above then yields our results.

To show that rows y_i and y_j where $j - i \geq 2$ do not overlap (i.e. share non-zero entries in the same spots), one must be a little more careful. Supposing that $a_i \neq 2$ (i.e. that $a_i > 2$), one uses the maximiser

$$K = (1, b_2, \dots, b_{i-2}, -a_{i-1}, a_i, -a_{i+1}, b_{i+2}, \dots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \dots, b_\ell),$$

which, by pushing-down at v_1 to the right and at v_{j-1} to the left, can be obtained from

$$(1, -b_2, \dots, -b_{i-2}, -b_{i-1}, a_i - 4, -b_{i+1}, -b_{i+2}, \dots, -b_{j-2}, a_{j-1}, a_j - 2, b_{j+1}, \dots, b_\ell) \in \mathcal{K}.$$

If instead $a_i = 2$ and there is some $k \in \{2, \dots, i-1\}$ such that $a_k \neq 2$, then we use

$$K = (1, b_2, \dots, b_{i-1}, a_i, -a_{i+1}, b_{i+2}, \dots, b_{j-2}, b_{j-1}, a_j, b_{j+1}, \dots, b_\ell),$$

a push-down of $(1, b_2, \dots, b_{i-1}, a_i - 2, -b_{i+1}, -b_{i+2}, \dots, -b_{j-2}, a_{j-1}, a_j - 2, b_{j+1}, \dots, b_\ell) \in \mathcal{K}$. Finally, if $a_k = 2$ for all $k = 2, \dots, i$, the fact that $y_i \cdot y_j = 0$ implies that if y_j and y_i overlap, then they overlap in two places. Consequently, since $y_{i-1} \cdot y_i = -1$, it follows that y_j also overlaps with y_{i-1} , and hence as $y_{i-1} \cdot y_j = 0$, that y_j overlaps in two places with y_{i-1} . Iterating this, we find eventually that y_j and x overlap, violating the condition $x \cdot y_j = 0$, since x contains precisely one non-zero entry (as $x^2 = 1$). Hence, y_i and y_j cannot overlap.

To show that y_i and y_{i+1} have only one overlap (in which they are opposite in sign), one uses

$$K = (1, b_2, \dots, b_{i-2}, -a_{i-1}, a_i, a_{i+1} - 2, b_{i+2}, \dots, b_\ell),$$

which is a push-down of $(1, -b_2, \dots, -b_{i-2}, -b_{i-1}, a_i - 2, a_{i+1} - 2, b_{i+2}, \dots, b_\ell) \in \mathcal{K}$.

Because $x^2 = n = 1$, the rest of the computation is trivial, and the theorem is proved.

4.3. Proof of Theorem 1.2 when $q = 1$. This last proof is even easier than in the previous section. Since none of the rows y_i exist, we need only prove the statement about x ; in the absence of the other rows, the only adjustments we need make to the proof of the changemaker statement are to define instead

$$J := \{ \langle K, [v_1] \rangle \mid K \in \mathcal{M}, D^{p/q}([K]) = 0 \},$$

and remove the assumption that $\sigma_0 = 1$. Once this is done, the modified statement follows easily.

4.4. A Remark on Vanishing Deficiencies. In its current form, the reader will hopefully have noticed the asymmetry in Theorem 1.2 concerning the number of deficiencies which vanish. If n is odd, we only require one to vanish, but if n is even, then we require $q - r + 1$. It is possible that by choosing a different function \mathfrak{r} we can remove this asymmetry, but as of the current writing we have been unable to do so.

What we can say, however, is that in the special case when $q = 2$, some simplifications are possible (c.f. [2, 8]). In practice, the following proposition is most readily applied when \mathfrak{t} is the unique Spin-structure.

Proposition 4.4. *In the case $q = 2$, Theorem 1.2 applies if we use a weaker assumption on the number of vanishing deficiencies. Namely, if n is even, we require only that*

$$d(Y, \mathfrak{t}) - d(S_{-p/q}^3(U), \mathfrak{t}) = 0,$$

for some $\mathfrak{t} \in \text{Spin}^c(Y)$.

Proof. When $q = 2$, notice that $p/q = [n, 2]$. We relabel row y_2 as y for convenience, and set $y = (1, 1, 0, \dots, 0)$ without loss of generality. Then $x \cdot y = -1$ tells us that

$$x_1 + x_2 = -1. \tag{17}$$

We let $x_2 \leq x_1$, also without loss of generality. Observe that $x_1 \geq 0$, else $x_1 + x_2 \leq -2$.

Now define a set

$$\mathcal{S} = \{K \in \mathcal{M} \mid \langle K, [v_2] \rangle = 0, D^{p/q}([K]) = 0\},$$

and observe that the maximal value j_{\max} of $\langle K, [v_1] \rangle$ obtained by letting K range over \mathcal{S} satisfies $j_{\max} \geq 0$, since we know that at least one deficiency vanishes. If $j_{\max} > 0$, however, then this means that at least $q - r + 1$ deficiencies vanish, and Theorem 1.2 applies. Thus, suppose $j_{\max} = 0$. By arguments similar to those in Section 4.1, we find that

$$j_{\max} = x_1 - x_2 + \sum_{i \geq 3} |x_i| = 0,$$

and on substituting from (17),

$$2x_1 + 1 + \sum_{i \geq 3} |x_i| = 0.$$

Since none of the terms on the LHS are negative, we have a contradiction. Hence $j_{\max} \neq 0$, and Theorem 1.2 applies. \square

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